Effect of combined triangularity and ellipticity on the stability limit of the ideal internal kink mode in a tokamak

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The sawtooth period and amplitude in tokamaks is known to depend strongly on the shape of the $q = 1$ surface [1]. Furthermore, this shape dependence seems to correlate, at least to some extent, with the stability limit of the ideal, internal kink mode. The strong dependence on shaping, especially on ellipticity $e$ and triangularity $\delta$, of the stability limit of this mode is well-known from numerical computations [2]. The strongly destabilizing effect of ellipticity alone, especially at small $\Delta q = 1 - q_0$, has recently also been given an analytical explanation by including terms of order $\varepsilon^2 e$, where $\varepsilon$ is the inverse aspect ratio, in a perturbation expansion of the potential energy $\delta W$ of the ideal $m = n = 1$ mode [3]. In the case of a parabolic current profile near the axis, and for small values of $\Delta q$ and $r_1$, a normalized form of this contribution to $\delta W$ is given by [3]

$$\delta \hat{W}^{(e\varepsilon)} = -\frac{3}{4}(\kappa_1 - 1)\beta_{p,1} + \frac{1}{2} \Delta q(\kappa_1 - 1)\left(13\beta_{p,1}^2 - \frac{1}{4}\beta_{p,1} - 1\right) + O(\Delta q^2). \quad (1)$$

Here, $\kappa_1$ denotes the elongation and $\beta_{p,1}$ the poloidal beta value at the $q = 1$ radius $r = r_1$. The normalization of $\delta W$ in Eq. (1) is such that the usual Bussac [4] expression for the potential energy, using the same normalization, current profile, and $\Delta q$-expansion, reads

$$\delta \hat{W}^{(e\varepsilon)} = \delta \hat{W}_{\text{Bussac}} = \Delta q \left(\frac{13}{48} - 3\beta_{p,1}^2\right) + O(\Delta q^2). \quad (2)$$

The stability condition for the ideal, internal kink mode in a toroidal plasma with circular cross section and small $q = 1$ radius is then given by the well-known pressure limit $\beta_{p,1} < \beta_{p,1}^{\text{crit}} = (13)^{1/2} / 12 \approx 0.3$. For a plasma with elliptical cross section, however, it is apparent that, due to the destabilizing (for $\kappa_1 > 1$) $\Delta q$-independent term in $\delta \hat{W}^{(e\varepsilon)}$, the stability condition based on both of the terms above leads to much smaller values of $\beta_{p,1}^{\text{crit}}$. See Ref. [3] for more details on this destabilizing effect of ellipticity alone.

Geometrically, the expansion parameter $\varepsilon$ represents a $\cos \theta$ perturbation (radial shift) of the flux surfaces, whereas the ellipticity parameter $e = (\kappa - 1)/2$ represents a $\cos 2\theta$ shape...
perturbation of the same flux surfaces. It turns out that contributions to \( \delta W \) to various orders \( \epsilon^{\mu}e^{\nu} \) are found for integers \( \mu \) and \( \nu \) such that \( \langle (\cos\theta)^{\mu}(\cos2\theta)^{\nu} \rangle \neq 0 \), where \( \langle \rangle \) denotes averaging over the interval \( 0 \leq \theta \leq 2\pi \). Thus, the leading-order terms in \( \delta W \) are of order \( \epsilon^2 \) (Bussac term) and \( \epsilon^2 \). The latter term is, however, very small for small \( \Delta q \) [5], and is for this reason omitted here. The next-order term is given by the \( \epsilon^2 e \)-term in Eq. (1), followed by \( \epsilon^2 e^2 \), \( \epsilon^2 e^3 \), \( \epsilon^2 e^4 \), \( \epsilon^2 e^5 \), etc., which are all neglected here.

Including also triangularity \( \delta \) in the equilibrium, we similarly expect contributions to \( \delta W \) to appear to orders \( \epsilon^{\mu}e^{\nu}\delta^{\lambda} \), where \( \langle (\cos\theta)^{\mu}(\cos2\theta)^{\nu}(\cos3\theta)^{\lambda} \rangle \neq 0 \). Thus, apart from the term \( \delta^2 \), which we neglect here for the same reason as \( \epsilon^2 \) was neglected above [5], the leading-order term in this category of terms is given by \( \epsilon \delta e \). Provided that \( \delta \) is at least of the same order as \( \epsilon \), this term is potentially of the same importance as the \( \epsilon^2 e \) term in Eq. (1). In this paper we present a recent computer algebra calculation of this term [6], using similar methods as in Ref. [3].

Compared with the pure ellipticity problem analyzed in Ref. [3], the volume of the present calculation is approximately a factor two larger. For instance, the number of non-vanishing poloidal side-bands involved in the present problem is ten \( (m = -2, -1, 2, 3, 4 \text{ to orders } \epsilon, e, \epsilon, e \text{ and } \epsilon e, \text{ respectively, }) \), as compared with the six poloidal side-bands involved in the analysis in Ref. [3].

Using the same normalization, current profile, and \( \Delta q \)-expansion as in Eqs. (1) and (2), it is shown in Ref. [6] that

\[
\delta \hat{W}^{(oe)} = \frac{3(\kappa_1 - 1)\beta_{p,1}\delta_1}{2\epsilon_1} + \frac{\Delta q(\kappa_1 - 1)\delta_1}{12\epsilon_1} \left( 4\beta_{p,1} - 7 \right) + O(\Delta q^2),
\]

where \( \delta_1 \) and \( \epsilon_1 \) denote the triangularity and inverse aspect ratio, respectively, of the \( q = 1 \) surface. We see that, for \( \kappa_1 > 1 \), the leading-order triangularity effect is stabilizing for positive \( \delta_1 \) and destabilizing for negative \( \delta_1 \), whereas the second term shows the opposite behaviour, provided that \( \beta_{p,1} < 7/4 \). Adding \( \delta \hat{W}^{(e)} \) and \( \delta \hat{W}^{(oe)} \) in Eqs. (1) and (3), respectively, we obtain the leading-order potential energy for the ideal, internal kink mode in the form

\[
\delta \hat{W}^{(e)} + \delta \hat{W}^{(oe)} = -\frac{3}{4}(\kappa_1 - 1)\beta_{p,1} \left( 1 - \frac{2\delta_1}{\epsilon_1} \right) + O(\Delta q).
\]

Thus, for sufficiently large triangularity, \( \delta_1 > \epsilon_1/2 \), the stabilizing, combined triangle-ellipticity effect in Eq. (3) overcomes the destabilizing ellipticity effect in Eq. (1) and, in total, positive ellipticity turns into a stabilizing rather than a destabilizing effect. This dependene on ellipticity and triangularity is exactly the same as in the corresponding Mercier criterion, and therefore not too surprising. See, e.g., Ref. [2], or Eq. (6) in Ref. [1].

In the figures below we illustrate the stability boundaries \( \delta \hat{W} = 0 \), with \( \delta \hat{W} = \delta \hat{W}^{(e)} + \delta \hat{W}^{(e)} + \delta \hat{W}^{(oe)} \), including the \( O(\Delta q) \)-terms in Eqs. (1)-(3), for various combinations of the parameters involved. Figure 1a shows the critical \( \beta_{p,1} \) as a function of \( \kappa_1 \) for \( \Delta q = 0.01, 0.05, \ldots \)
0.1 and 0.2 and for $\delta_l/\varepsilon_1 = 0.4$. Thus, the triangularity is slightly below the limit ($\varepsilon_l/2$) above which the leading-order ellipticity term in Eq. (3) becomes stabilizing. Consequently, $\beta_{p,1}^{cu}$ is smaller than 0.3 for these parameters. Figure 1b shows the same type of diagram, but with $\delta_l/\varepsilon_1 = 0.6$, and the critical beta values accordingly larger than 0.3. In the case $\Delta q = 0.2$, however, the destabilizing triangularity effect in the second term in Eq. (3) can be seen. Figure 1c shows the stability boundary at the critical value $\delta_l/\varepsilon_1 = 0.5$. This stability boundary is independent of $\Delta q$ in the present approximation.

These stability boundaries are shown in a couple of other ways in Figs. 2 and 3 below. In Fig. 2a the critical $\beta_{p,1}$ is shown as a function of $\kappa_1$ for $\Delta q = 0.05$ and for different $\delta_l/\varepsilon_1$. The improvement of stability as $\delta_l/\varepsilon_1$ goes from 0 to 0.4 is seen to be moderate for all values of $\kappa_1$, whereas a dramatic improvement occurs when $\delta_l/\varepsilon_1$ increases from 0.4 to 0.6. This effect is also illustrated in Fig. 2b which shows the critical $\beta_{p,1}$ as a function of $\delta_l/\varepsilon_1$ for $\kappa_1 = 1.2$ and $\Delta q = 0.01, 0.05, 0.1$ and 0.2. For sufficiently small values of $\Delta q$, there is a dramatic improvement of the stability when $\delta_l/\varepsilon_1$ becomes larger than 0.5, as predicted by Eq. (4).
In Fig. 5 in Ref. [1], the stability threshold from the Mercier criterion is plotted as the maximum allowable elongation $\kappa_1$ for stability vs the triangularity $\delta_1$. In Figs. 3a,b below we illustrate the stability regions of the internal kink mode, and the stabilizing effect of triangularity, in a similar way. For instance, Fig. 3a shows the $\kappa_1$-limit as a function of $\delta_1/\epsilon_1$ for $\Delta q = 0.05$ and for different values of $\beta_{p,1}$. The regions of stability are below the solid lines (for all $\beta_{p,1}$), and above the dashed line (for $\beta_{p,1} = 0.4$ only). The diagram in Fig. 3b is similar, but here $\beta_{p,1}$ is fixed ($= 0.1$) whereas $\Delta q$ varies. Also here, the regions of stability are below the solid lines and above the dashed line (for $\Delta q = 0.01$).

In conclusion, we have investigated the effects of a combined triangular and elliptical plasma cross-section on the stability of the ideal, internal kink mode in a tokamak. The work extends the results of a preceding paper [3], where the effect of ellipticity alone was investigated. The full details of the present work can be found in a forthcoming paper [6]. The main result is an expression for the potential energy of the ideal, internal kink mode to order $\epsilon \epsilon_1$, given (in normalized form) by Eq. (3) in the case of a parabolic current profile near the axis, and for small values of $\Delta q$. When this contribution to the potential energy is added to the corresponding ellipticity term in $\delta W$, derived in Ref. [3], the combined expression, Eq. (4), assumes a similar form as the corresponding term in the Mercier criterion [2]. Thus, similarly to Mercier stability of shaped tokamaks we find in the case of the internal kink mode that, whereas ellipticity alone is destabilizing, a combination of ellipticity and positive triangularity is stabilizing. Both experimental observations of shaping effects on the sawtooth stability [1] and numerical computations [2] show similar dependences on ellipticity and triangularity.

References